Sample Final Math 98 N1 (on line)

Note
The exam will be closed book. A 4x6 double-sided hand-written note card and a calculator are allowed.
This course is about using tools like equations to solve problems. The advantage compared to hit and miss, or listing all possibilities until we hit the answer, is that we have a much faster and flexible way to answer questions, and we are able to tackle situations where the solutions involve really big numbers, that would not be easily manageable on a case by case basis (e.g., “how many years to reach $1,000,000?” the answer being 1245: if you went counting year by year, or even decade by decade, it would take a very long time to find the answer). Hence, in the following word problems, always set up the appropriate equations and solve them: do not try to go “year by year” (or what the case may be). If you do, your answer will get a score of 0.

1. Linear Functions
   1. Consider the function $f(x) = \frac{x}{2} - 3$
      
      1.1. Determine $f\left(\frac{4}{3}\right)$
      
      1.2. Solve the equation $f(x) = -2$
   
   2. A nest egg is set up to cover everyday expenses that average $120 a week. Suppose it starts at $2,500, with a plan for replenishing it once it falls below $200. After how many weeks will it need replenishing? Use equations and inequalities, not tables (which means calculating the balance week by week, until it falls below 200: such a solution will score 0)
   
   3. A reservoir holding 4,000 gallons of water is depleted at a constant rate of 250 gallons per hour. At the same time, an adjacent reservoir is being filled at a constant rate of 140 gallons per hour, starting from empty. At what time will the two hold the same amount of water, and how much is that amount? Use equations, not tables, exactly in the same spirit as described in the previous problem.

Solutions

1. 
   
   1.1. We plug in $\frac{4}{3}$ wherever we see $x$, while remembering that $\frac{x}{2}$ is exactly the same thing as $x \cdot \frac{1}{2}$. Hence, $\frac{1}{2} \cdot \frac{4}{3} - 3 = \frac{4}{6} - 3 = \frac{2}{3} - \frac{9}{3} = -\frac{7}{3}$
   
   1.2. We proceed by “solving for $x$“, as always when solving equations:
   
   $\frac{x}{2} - 3 = -2$, $\frac{x}{2} = 1$, $x = 2$

Do not solve problems like this by plugging in numbers as trial and error: it is an inefficient method, and it doesn’t work at all if the solution is not an integer or a very simple fraction. If you do that in an exam, your score on this type of problem will be 0, even if your solutions is correct.
2. We are looking for a linear function, and, measuring time in weeks, the downward (negative) slope is \(-120\), while the initial value (the “y-intercept”) is 2500. Hence, the equation for this nest egg is \(2500 - 120t\), if \(t\) is the number of weeks since the start. This line will cross the level 200 when \(2500 - 120t = 200\), after which it will be below this line. That will happen when \(-120t = 200 - 2500\), or \(t = \frac{-2300}{-120} = \frac{2300}{120} = \frac{115}{6} \approx 19.17\). That is sometime, shortly into the 20th week. Note that you can also cast this problem as an inequality, perhaps staying closer to the model itself: the nest egg is under water when \(2500 - 120t < 200\), that is \(-120t < -2300\). Dividing by \(-120\), a negative quantity, reverses the inequality; hence this happens when \(t > \frac{2300}{120} = \frac{115}{6}\).

3. The first reservoir starts at a level of 4000 (gallons), and is decreasing linearly, hence, with a negative slope of \(-250\) (note that this works if we agree to measure volume in gallons and time in hours. If you choose different units, say, liters, and/or days, the numbers change!). The equation is then \(R_1(t) = 4000 - 250t\). Similarly, the second reservoir starts at 0, and increases linearly (positive slope) at 140 gallons.hour, resulting in the equation \(R_2(t) = 140t\). The two are at the same level when \(R_1(t) = R_2(t)\), that is \(4000 - 250t = 140t\). Solving for \(t\),
\[
4000 = 390t
\]
\[
t = \frac{4000}{390} = \frac{400}{39} \approx 10.26
\]

2. Polynomials

1. Simplify as much as possible the following expression
\[
(x^2 - 2x - 1)(x - x^3) - (2 - x^2)^2
\]

2. A business expects to sell \(w\) widgets (an imaginary item) if their selling price is \(p\), according to an equation of the form \(w = ap + b\). Knowing that we expect to sell 100 widgets if the price is $5, and only 20 if the price is $15, find \(a\) and \(b\) in the equation.

2.2. According to this model, if the price is set at \(p\), we will collect \(pw\) dollars from our sales. Determine the price(s) at which we would collect $0 (either because we give stuff away for free, or because nobody is buying our widgets at that price)
Solutions

1. We have to respect the order of operations: first powers, then products, last sums, unless overruled by parentheses. In our case, that leads to
\[ x^3 - x^5 - 2x^2 + 2x^4 - x + x^3 - (4 - 4x^2 + x^4) \] (using the distributive property for the first product and shortcutting the square in the second term using the “special product” \((a - b)^2 = a^2 - 2ab + b^2\), which you can check by distributing \((a - b) \cdot (a - b)\)).

Moving on, we find
\[-x^5 + 2x^4 + 2x^3 - 2x^2 - x - 4 + 4x^2 - x^4 = -x^5 + x^4 + 2x^3 + 2x^2 - x - 4\]

2. 2.1. We are simply looking at the equation\(^1\) of a line going through the two points (using “price” as input, and “sales” as output) \((5, 100)\) and \((15, 20)\). For example, we can compute the slope, \(\frac{20 - 100}{15 - 5} = \frac{-80}{10} = -8\), and make sure the line goes through, for example, \((15, 20); 20 = -8 \cdot 15 + b\), that is \(b = 20 + 120 = 140\). The equation is then \(w(p) = -8p + 140\).

2.2. Since each sale brings in \(p\), when the price is set at \(p\), the total cash collected\(^2\) will be \(pw(p) = p(140 - 8p) = 140p - 8p^2\). This function will be zero when \(p(140 - 8p) = 0\). As this equation is already in factored form, we can apply the “zero-product property”, and see that there are two solutions: \(p = 0\) (corresponding to giving our widgets away for free, which, of course, will not bring any revenue), and \(140 - 8p = 0\), that is \(p = \frac{140}{8} = 17.50\) (corresponding to a price so high that nobody will buy our widgets,

3. Exponential and Logarithmic Functions

1. Determine an exponential function going through the points \((-1, 2)\) and \((\frac{3}{2}, \frac{1}{3})\)

2. Compound interest works as follows: an investment (“principal”) of \(P\) at an annual interest rate of \(r\), will be worth, after \(t\) years, \(P (1 + r)^t\) (compounding may occur more than once a year—in fact, that’s the case with most loans—but we are thinking here of \(r\) being the Annual Percentage Yield (APY), that is how much your principal will yield after 12 months, under whatever compounding schedule you are under). For example, suppose you took out a $1,000 loan at 5% APY. After 3.25 years (three years and three months), your liability will have increased to \((P = 1000, r = 0.05, t = 3.25)\)
\[1000 \cdot (1 + 0.05)^{3.25} = 1000 \cdot 1.05^{3.25} = 1000 \cdot 1.17183 = 1171.83\text{ dollars}.\]

2.1. Suppose you invest $250 at an APY of 1%. How long will it take to grow to $300?

2.2. Your friend has invested only $100 at the same time, but her APY is 2.5%. When will her investment’s value overtake yours?

\(^1\) A function such as this is called by economists a demand function. A linear form is a bit simplistic, but it is simple and allows for rough estimates, while more sophisticated forms might not provide sufficient improvement to justify the complexity.

\(^2\) The resulting function is called a revenue function
Solutions

1. Looking for a function of the form $f(x) = ab^x$, such that $f(-1) = 2$, $f\left(\frac{3}{2}\right) = \frac{1}{2}$, we need numbers $a$ and $b > 0$, such that $ab^{-1} = 2$, $ab^{\frac{3}{2}} = \frac{1}{2}$. One way to solve this problem is to use the first equation to observe that we must have $\frac{a}{b} = 2 \Rightarrow a = 2b$. Plugging this into the second gives us $2b \cdot b^{\frac{3}{2}} = \frac{1}{2}$. Using the properties of exponents, we then find $b^{1+\frac{3}{2}} = b^{\frac{5}{2}} = \frac{1}{2}$. That implies $a = 2b = 2 \cdot \left(\frac{1}{4}\right)^{\frac{3}{2}}$. Combining the two, we find

$$f(x) = 2 \cdot \left(\frac{1}{16}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{16}\right)^{\frac{3}{2}} = 2 \cdot \left(\frac{1}{16}\right)^{\frac{1}{2} \cdot \frac{3}{2}}.$$  Since $2^5 = 32 = 2 \cdot 16$, this can also be written as $2^\frac{1}{2} \cdot \left(\frac{1}{16}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{16}\right)^{\frac{3}{2}} = 2^{\frac{1}{2} - \frac{3}{2}} = 2^{\frac{1}{2} - \frac{3}{2}}$, with more variations possible (any of these formulas is correct, of course).

2. 2.1. We only need to plug in the numbers in the formula to find our equation:

$$250 \cdot (1 + 0.01)^t = 250 \cdot 1.01^t = 300,$$  where $t$ is the unknown we are looking for.

To solve $1.01^t = \frac{300}{250} = \frac{6}{5} = 1.2$ we have a few (completely equivalent) methods. One is to “take logarithms” of both sides. It doesn’t matter which base we use, but to get numerical answer we are restricted to the bases that are available on our calculators, $10$ and $e$. Sticking with $10$, we have

$$t \log(1.01) = \log(1.2) \Rightarrow t = \frac{\log(1.2)}{\log(1.01)} \approx 18.323.$$  Another way of doing the same thing is to write $1.01 = 10^{\log(1.01)}$, so the equation reads as $10^{\log(1.01)t} = 1.2$, that is $\log(1.01)t = \log(1.2)$, for the same result.

2.2. Your friend’s investment is described in the same way, with the appropriate numbers: $P = 100$, $r = 0.025$. After $t$ years, it will grow to $100 \cdot 1.025^t$. This will catch up with your investment when $100 \cdot 1.025^t = 250 \cdot 1.01^t$. Dividing by 100 both sides, and dividing by $1.01^t$ both side as well, this equation changes to

$$\frac{1.025^t}{1.01^t} = 2.5.$$  Using the properties of powers, that’s the same as $\left(\frac{1.025}{1.01}\right)^t = 2.5$.

This is the same type of equation that we solved in point 2.1, and can be solved in the same way. For example, taking logarithms of both sides, we have the solution

$$t = \frac{\log(2.5)}{\log\left(\frac{1.025}{1.01}\right)} = \frac{\log(2.5)}{\log(1.025) - \log(1.01)}.$$  The first expression is probably better in terms of rounding errors, but, in practice, it will make little difference, even if, instead, we used the fact that $\frac{1.025}{1.01} \approx 1.01485$, and solved $1.01485^t = 2.5$. In any case, using a calculator or software, we find the (approximate) solution

$$t \approx 65.154.$$  That’s a long time, an example of how a higher rate of return will always allow you to catch up with a higher initial investment, but the time required might be considerable.

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3 If this was an exam, and you didn’t have scientific calculator, you could simply stop at the solution formula for $t$, which is the actual answer that is asked for.